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On Hyperbolic Splines

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1. INTRODUCTION

In recent years there has been an intense interest in spline functions (see [4] and references therein). Among the various classes of splines, the polynomial spline functions have received by far the greatest attention, primarily because they are the most useful in numerical computations. This, in turn, is due largely to the fact that the polynomial splines admit a basis of so-called B-splines which can be computed efficiently and accurately via certain recursion relations. Recently (see [3]) it was discovered that certain classes of trigonometric splines also admit of B-spline bases which satisfy similar recursion relations.

The purpose of this paper is to give a detailed discussion of a third class of splines, the hyperbolic splines, which also have a basis of B-splines which can be computed recursively. In addition to their value in certain applications and as an illustration of the space of L-splines (cf. [4]), the hyperbolic splines are of special interest in view of the fact (see [5]) that the only classes of splines which have B-spline bases computable by recursions are the polynomial, trigonometric, and hyperbolic splines.

Our treatment of hyperbolic splines depends heavily on obtaining explicit formulae for a related Green's function, for determinants formed from the hyperbolic functions, and for certain associated hyperbolic divided differences. These results are developed in Sections 2–4. The hyperbolic B-splines are introduced in Section 5, and the key recursion relation is established in Section 6. In the remaining sections of the paper we discuss the shape of the B-splines, a Peano kernel representation for divided differences, integrals of the B-splines, a Marsden-type identity, a partition of unity result, and, finally, give a basis and dual basis for $\mathscr{S}$.

We turn now to the definition of hyperbolic splines. First we need some notation. Throughout the paper we shall use the abbreviations

$$\mathrm{ch}(x) = \cosh(x), \qquad \mathrm{sh}(x) = \sinh(x).$$

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Given any nonnegative integer r, we define the sets

$$V_m = \{v_1,..., v_m\} = \{\mathrm{ch}(x), \mathrm{sh}(x),..., \mathrm{ch}((2r-1)\,x), \mathrm{sh}((2r-1)\,x)\}, \qquad m = 2r$$

$$U_m = \{u_1,..., u_m\} = \{1, \mathrm{sh}(2x), \mathrm{ch}(2x),..., \mathrm{sh}(2rx), \mathrm{ch}(2rx)\}, \qquad m = 2r+1.$$

Associated with these sets we define the linear space

$$\begin{aligned} \mathscr{H}_m &= \mathrm{span}(V_m), && m = 2r, \\ &= \mathrm{span}(U_m), && m = 2r+1. \end{aligned}$$

This space is the null space of the differential operator

$$\begin{aligned} L_m &= (D^2-(2r-1)^2)\cdots(D^2-3^2)(D^2-1), && m = 2r, \\ &= (D^2-(2r)^2)\cdots(D^2-2^2)D, && m = 2r+1. \end{aligned} \tag{1.1}$$

Suppose $\Delta = \{a = x_0 < x_1 < \cdots < x_{k+1} = b\}$ is a partition of $[a, b]$ and that $\mathscr{M} = (m_1,..., m_k)$ is a vector of positive integers with $m_i \leqslant m$, $i = 1, 2,..., k$. Then we call

$$\begin{aligned} \mathscr{S}(\mathscr{H}_m; \mathscr{M}; \Delta) = \{s\colon &\text{there exists } s_0,..., s_k \in \mathscr{H}_m \text{ with } s(x) \mid (x_i, x_{i+1}) = s_i \\ &i = 0, 1,..., k \text{ and } D^{j-1}s_{i-1}(x_i) = D^{j-1}s_i(x_i), \\ &j = 1, 2,..., m-m_i \text{ and } i = 1, 2,..., k\} \end{aligned} \tag{1.2}$$

the *space of hyperbolic splines of order m with knots $x_1,..., x_k$ of multiplicities $m_1,..., m_k$*.

The space $\mathscr{S}$ is a space of L-splines—see [4, Chapter 10]. By the general theory (cf. [4, p.430 and Theorem 4.4]), we know that $\mathscr{S}$ is a linear space of dimension $m + K$ with $K = \sum_1^k m_i$. We begin our detailed examination of this space of L-splines by giving an explicit formula for the Green's function associated with the operator L_m.

2. The Green's Function

The aim of this section is to show that the function

$$G_m(x; y) = ((x-y)_+^0/(m-1)!)[\mathrm{sh}(x-y)]^{m-1} \tag{2.1}$$

is the Green's function associated with the operator L_m and appropriate initial conditions. We begin with a lemma which gives a useful expansion for powers of the hyperbolic sine.

LEMMA 2.1. *Given any nonnegative integer r,*

$$[\mathrm{sh}(\theta)]^{m-1} = \frac{(-1)^r}{2^{m-2}} \sum_{\nu=1}^{r} (-1)^\nu \binom{2r-1}{r+\nu-1} \mathrm{sh}((2\nu-1)\,\theta), \qquad m=2r,$$
$$= \frac{(-1)^r}{2^{m-2}} \left[\binom{2r-1}{r} + \sum_{\nu=1}^{r} (-1)^\nu \binom{2r}{r+\nu} \mathrm{ch}(2\nu\theta)\right], \quad m=2r+1. \tag{2.2}$$

Proof. The result is obvious for $m=1$ and $m=2$. It then follows for general m by a straightforward inductive argument. ∎

As an immediate consequence of this lemma and elementary identities for the hyperbolic functions, we have the following expansion for the kernel appearing in (2.1):

$$\begin{aligned}
&[\mathrm{sh}(y-x)]^{m-1} \\
&\quad = \frac{(-1)^r}{2^{m-2}} \sum_{\nu=1}^{r} (-1)^\nu \binom{2r-1}{r+\nu-1} [\mathrm{sh}((2\nu-1)\,y) \\
&\quad\quad \times \mathrm{ch}((2\nu-1)\,x) - \mathrm{ch}((2\nu-1)\,y)\,\mathrm{sh}((2\nu-1)\,x)], \qquad m=2r, \\
&\quad = \frac{(-1)^r}{2^{m-2}} \left\{ \sum_{\nu=1}^{r} (-1)^\nu \binom{2r}{r+\nu} [\mathrm{ch}(2\nu y)\,\mathrm{ch}(2\nu x) \right. \\
&\quad\quad \left. - \mathrm{sh}(2\nu y)\,\mathrm{sh}(2\nu x)] + \binom{2r-1}{r} \right\}, \qquad m=2r+1.
\end{aligned} \tag{2.3}$$

This shows that for each fixed y, the function $[\mathrm{sh}(y-x)]^{m-1}$ belongs to the space $\mathscr{H}_m$.

We can now show that G_m is a Green's function associated with L_m.

THEOREM 2.2. *For any positive integer m,*

$$L_m G_m(x;y) = 0 \qquad \textit{for all } x \neq y \quad (\textit{where } L_m \textit{ operates on the x-variable}), \tag{2.4}$$

$$D_x^j G_m(x;y)\,|_{y=x} = \delta_{j,m-1}, \qquad j=0,1,\dots,m-1. \tag{2.5}$$

Proof. It is clear from Lemma 2.1 and the definition of G_m that for each fixed y, $G_m(x;y)$ is in $\mathscr{H}_m$, and thus that $L_m G_m(x;y)=0$ for all $x \neq y$. To prove (2.5), we apply D_x^j to the definition of G_m to obtain

$$\frac{D_x^j[\mathrm{sh}(x-y)]^{m-1}}{(m-1)!} = \frac{[\mathrm{sh}(x-y)]^{m-j-1}}{(m-j-1)!}\,[\mathrm{ch}(x-y)]^j + \cdots,$$

where each of the other terms contains a power of $\mathrm{sh}(x-y)$. It follows that for $0 \leqslant j \leqslant m-2$, $D_x^j G_m(x;y)|_{y=x} = 0$. Now if $j = m-1$, then

$$D_x^{m-1} G_m(x;y)|_{y=x} = [\mathrm{ch}(x-y)]^{m-1} + \cdots,$$

where each of the other terms includes some power of $\mathrm{sh}(y-x)$, and evaluating at $y = x$ gives the value 1. ∎

The Green's function G_m defined in (2.1) will play an essential role in defining a B-spline basis for the space of hyperbolic splines. The following theorem (which follows immediately from general results on L-splines—see, [4, Theorem 10.8]) shows that it is also the kernel for a useful generalized Taylor expansion. We use the notation

$$L_1^m[a,b] = \{f\colon D^{m-1}f \text{ is absolutely continuous on } [a,b] \text{ and } D^m f \in L_1[a,b]\}.$$

THEOREM 2.3 (Taylor expansion). *Let $f \in L_1^m[a,b]$. Then*

$$f(x) = u_f(x) + \int_a^b G_m(x;y)\, L_m f(y)\, dy, \qquad \text{all} \quad a \leqslant x \leqslant b, \tag{2.6}$$

where u_f is the unique element in $\mathscr{H}_m$ = null space of L_m such that

$$D^{j-1}u_f(a) = D^{j-1}f(a), \qquad j = 1, 2,..., m.$$

3. Some Basic Determinants

Our main tool for constructing a B-spline basis for the space of hyperbolic splines defined in (1.2) will be certain hyperbolic divided differences. Before defining these divided differences, we need to introduce some determinants associated with the functions $\{u_1,..., u_m\}$ and $\{v_1,..., v_m\}$ spanning the space $\mathscr{H}_m$.

Given any points $t_1 < t_2 < \cdots < t_m$, we define

$$D\begin{pmatrix} t_1,..., t_m \\ u_1,..., u_m \end{pmatrix} = \begin{vmatrix} u_1(t_1) & u_2(t_1) & \cdots & u_m(t_1) \\ u_1(t_2) & u_2(t_2) & \cdots & u_m(t_2) \\ \vdots & & & \\ u_1(t_m) & u_2(t_m) & \cdots & u_m(t_m) \end{vmatrix}.$$

We extend the definition of this determinant to the case $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_m$ in the usual way (cf. [4, p. 21]). Determinants formed from the $v_1,..., v_m$ are defined analogously.

The following theorem established an important property of these determinants:

THEOREM 3.1. *For any integer* $1 \leqslant k$,

$$D\begin{pmatrix} t_1,\ldots,t_k \\ u_1,\ldots,u_k \end{pmatrix} > 0 \qquad \textit{for all } t_1 \leqslant t_2 \leqslant \cdots \leqslant t_k$$

while

$$D\begin{pmatrix} t_1,\ldots,t_k \\ v_1,\ldots,v_k \end{pmatrix} > 0 \qquad \textit{for all } t_1 \leqslant t_2 \leqslant \cdots \leqslant t_k.$$

In other words, $U_m = \{u_i\}_1^m$ *and* $V_m = \{v_i\}_1^m$ *are Extended Complete Tschebyscheff systems* (cf. [2, 4]).

Proof. We give the proof for the u's—the proof for the v's is analogous. By a well-known theorem from the theory of Tschebyscheff systems (see [2, p. 377]), it suffices to show that the Wronskian determinants formed from the u's satisfy

$$W(u_1,\ldots,u_k)(x) > 0 \qquad \text{for all real } x \text{ and all } k = 1, 2,\ldots. \tag{3.1}$$

We accomplish this by induction on k. For $k = 1$ we have $W(u_1)(x) = 1$. Now suppose that (3.1) has been established for $k - 1$—we now show that it holds for k. If $k = 2r$ is even, we must examine the determinant

$$W(u_1,\ldots,u_k)(x) = \begin{vmatrix} 1 & \operatorname{sh}(2x) & \operatorname{ch}(2x) & \cdots & \operatorname{sh}(2rx) \\ 0 & 2\operatorname{ch}(2x) & 2\operatorname{sh}(2x) & \cdots & (2r)\operatorname{ch}(2rx) \\ \vdots & & & & \\ 0 & 2^{2r-1}\operatorname{ch}(2x) & 2^{2r-1}\operatorname{sh}(2x) & \cdots & (2r)^{2r-1}\operatorname{ch}(2rx) \end{vmatrix}.$$

Consider the linear system

$$\begin{bmatrix} 1 & 2^2 & 2^4 & \cdots & 2^{2r-2} \\ 1 & 4^2 & 4^4 & \cdots & 4^{2r-2} \\ \vdots & & & & \\ 1 & (2r)^2 & (2r)^4 & \cdots & (2r)^{2r-2} \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \alpha_4 \\ \vdots \\ \alpha_{2r} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Since the matrix of this system (as well as the $r - 1$ by $r - 1$ minor in the upper left-hand corner) is a VanderMonde matrix, we can uniquely solve this system for $\alpha_2, \alpha_4, \ldots, \alpha_{2r}$. By Cramer's rule, we see that $\alpha_{2r} > 0$. Now if we multiply the second row of W by α_2/α_{2r}, the fourth row by $\alpha_4/\alpha_{2r}, \ldots,$ and the $(2r-2)$th row by $\alpha_{2r-2}/\alpha_{2r}$ and add these numbers to the last row, we convert W to a new determinant (with the same value) whose last row is

$$[0 \quad 0 \quad \cdots \quad 0 \quad (2r/\alpha_{2r})\,\mathrm{ch}(2rx)].$$

But then by the induction assumption,

$$W(u_1, \ldots, u_k)(x) = (2r/\alpha_{2r})\; W(u_1, \ldots, u_{k-1})(x) > 0.$$

The analysis in the case where $k = 2r+1$ is odd proceeds somewhat differently. In this case we have

$$W(u_1, \ldots, u_k)(x) = \begin{vmatrix} 1 & \mathrm{sh}(2x) & \mathrm{ch}(2x) & \cdots & \mathrm{ch}(2rx) \\ 0 & 2\,\mathrm{ch}(2x) & 2\,\mathrm{sh}(2x) & \cdots & (2r)\,\mathrm{sh}(2rx) \\ \vdots & & & & \\ 0 & 2^{2r}\,\mathrm{sh}(2x) & 2^{2r}\,\mathrm{ch}(2x) & \cdots & (2r)^{2r}\,\mathrm{ch}(2rx) \end{vmatrix}.$$

In this case we combine the rows $2, 4, \ldots, 2r$ and then the rows $3, 5, \ldots, 2r+1$ to reduce W to the form

$$W = \left|\begin{array}{cccc|cc} 1 & \mathrm{sh}(2x) & \mathrm{ch}(2x) & \cdots & & \\ 0 & 2\,\mathrm{ch}(2x) & 2\,\mathrm{sh}(2x) & \cdots & & \\ \vdots & & & & & \\ 0 & 2^{2r-2}\,\mathrm{sh}(2x) & 2^{2r-2}\,\mathrm{ch}(2x) & \cdots & & \\ \hline 0 & 0 & \cdots & 0 & \mathrm{ch}(2rx) & \mathrm{sh}(2rx) \\ 0 & 0 & \cdots & 0 & \mathrm{sh}(2rx) & \mathrm{ch}(2rx) \end{array}\right|.$$

Expanding by the Laplace expansion and using the inductive hypothesis together with the fact that the 2×2 determinant in the corner is

$$\begin{vmatrix} \mathrm{ch}(2rx) & \mathrm{sh}(2rx) \\ \mathrm{sh}(2rx) & \mathrm{ch}(2rx) \end{vmatrix} = 1,$$

show that $W > 0$. This completes the proof for the u's. The proof for the v's is nearly the same. ∎

Theorem 3.1 shows that the determinants formed from the u's and from the v's are always positive. Our next result gives explicit formulae for these determinants in the case the t's are distinct.

THEOREM 3.2. *For any integer $r > 0$ and any $t_1 < t_2 < \cdots < t_m$,*

$$D\begin{pmatrix} t_1,\ldots,t_m \\ u_1,\ldots,u_m \end{pmatrix} = 2 \times 4^{r^2-2}\,\mathrm{ch}(t_1 + \cdots + t_m) \prod_{1 \leqslant i<j \leqslant m} \mathrm{sh}(t_j - t_i), \quad m = 2r, \tag{3.2}$$

$$D\begin{pmatrix} t_1,\ldots,t_m \\ v_1,\ldots,v_m \end{pmatrix} = 4^{r^2-r} \prod_{1 \leqslant i<j \leqslant m} \mathrm{sh}(t_j - t_i), \quad m = 2r, \tag{3.3}$$

$$D\begin{pmatrix} t_1,\ldots,t_m \\ u_1,\ldots,u_m \end{pmatrix} = 4^{r^2} \prod_{1 \leqslant i<j \leqslant m} \mathrm{sh}(t_j - t_i), \quad m = 2r+1, \tag{3.4}$$

$$D\begin{pmatrix} t_1,\ldots,t_m \\ v_1,\ldots,v_m \end{pmatrix} = 4^{r^2}\,\mathrm{ch}(t_1 + \cdots + t_m) \prod_{1 \leqslant i<j \leqslant m} \mathrm{sh}(t_j - t_i), \quad m = 2r+1. \tag{3.5}$$

Proof. The proof proceeds by induction. The values of all determinants are easily computed in the cases of $m = 1$ and 2. Suppose now that the results are valid for determinants of size $m - 1$. We now establish them for determinants of size m. We begin with the determinant in (3.2). First, we note that by using (3.4) for $m - 1$, we have

$$\begin{aligned} D(x) = D\begin{pmatrix} t_1,\ldots,t_{m-1},x \\ u_1,\ldots,u_m \end{pmatrix} &= \mathrm{sh}(2rx) \cdot D\begin{pmatrix} t_1,\ldots,t_{m-1} \\ u_1,\ldots,u_{m-1} \end{pmatrix} + \sum_{i=1}^{m-1} a_i u_i(x) \\ &= \mathrm{sh}(2rx) 4^{(r-1)^2} \prod_{1 \leqslant i<j \leqslant m-1} \mathrm{sh}(t_j - t_i) + \sum_{i=1}^{m-1} a_i u_i(x) \end{aligned}$$

for some coefficients $a_1,\ldots,a_{m-1}$. Clearly, $D(t_i) = 0$, $i = 1, 2,\ldots, m - 1$. On the other hand, using Lemma 3.3 below and a standard hyperbolic identity, we see that

$$C(x) = \mathrm{ch}(x + t_1 + \cdots + t_{m-1}) \prod_{i=1}^{m-1} \mathrm{sh}(x - t_i) = \frac{\mathrm{sh}(2rx)}{2^{m-1}} + \sum_{i=1}^{m-1} b_i u_i(x)$$

(where $b_1,\ldots,b_{m-1}$ are certain coefficients) also vanishes at the same points. Now since by Theorem 3.1 the set $\{u_i\}_1^m$ forms an Extended Tschebyscheff system, we conclude that $D(x)$ must be a constant multiple of $C(x)$, and comparing the coefficients of $\mathrm{sh}(2rx)$, we conclude that

$$D(x) = 2^{m-1} 4^{r^2-2r+1} \operatorname{ch}(x + t_1 + \cdots + t_{m-1}) \times \prod_{1 \leqslant i < j \leqslant m-1} \operatorname{sh}(t_j - t_i) \prod_{i=1}^{m-1} \operatorname{sh}(x - t_i).$$

Evaluating this expression at $x = t_m$, we obtain (3.2).

The proofs of the other determinant formulae are similar. To get (3.3), we note that by (3.5) for $m-1$, we have

$$D(x) = D\begin{pmatrix} t_1, \dots, t_{m-1}, x \\ v_1, \dots, v_m \end{pmatrix} = \operatorname{sh}((2r-1)x)\, 4^{(r-1)^2} \operatorname{ch}(t_1 + \cdots + t_{m-1}) \times \prod_{1 \leqslant i < j \leqslant m-1} \operatorname{sh}(t_j - t_i) + \sum_{i=1}^{m-1} a_i v_i(x).$$

We compare this with the function

$$C(x) = \prod_{i=1}^{m-1} \operatorname{sh}(x - t_i) = \frac{\operatorname{sh}((2r-1)x)}{2^{m-2}} \operatorname{ch}(t_1 + \cdots + t_{m-1}) + \sum_{i=1}^{m-1} b_i v_i(x).$$

Clearly both $D(x)$ and $C(x)$ vanish at the same set of points $t_1, \dots, t_{m-1}$ and since $\{v_i\}_1^m$ is an Extended Tschebyscheff system by Theorem 3.1, we conclude that $D(x)$ is a constant multiple of $C(x)$. Comparing coefficients of $\operatorname{sh}((2r-1)x)$, we conclude that

$$D(x) = 2^{m-2} 4^{(r-1)^2} \prod_{1 \leqslant i < j \leqslant m-1} \operatorname{sh}(t_j - t_i) \prod_{i=1}^{m-1} \operatorname{sh}(x - t_i),$$

and evaluating this expression at $x = t_m$ yields (3.3).

To establish (3.4), we note that by (3.2) for $m-1$, we have

$$D(x) = D\begin{pmatrix} t_1, \dots, t_{m-1}, x \\ u_1, \dots, u_m \end{pmatrix} = 2 \operatorname{ch}(2rx)\, 4^{r^2-r} \operatorname{ch}(t_1 + \cdots + t_{m-1}) \times \prod_{1 \leqslant i < j \leqslant m-1} \operatorname{sh}(t_j - t_i) + \sum_{i=1}^{m-1} a_i u_i(x)$$

while

$$C(x)=\prod_{i=1}^{m-1}\operatorname{sh}(x-t_i)=\operatorname{ch}(2rx)\frac{\operatorname{ch}(t_1+\cdots+t_{m-1})}{2^{m-2}}+\sum_{i=1}^{m-1} b_i u_i(x).$$

This implies that

$$D(x)=2\cdot 2^{m-2}4^{r^2-r}\prod_{1\leqslant i<j\leqslant m-1}\operatorname{sh}(t_j-t_i)\prod_{i=1}^{m-1}\operatorname{sh}(x-t_i),$$

and evaluating at $x=t_m$ yields (3.4).

Finally, to establish (3.5), we use (3.3) for $m-1$ to obtain

$$D(x)=D\begin{pmatrix} t_1,\ldots,t_{m-1},x \\ v_1,\ldots,v_m\end{pmatrix}=\operatorname{ch}(mx)\,4^{r^2-r}\prod_{1\leqslant i<j\leqslant m-1}\operatorname{sh}(t_j-t_i)$$
$$+\sum_{i=1}^{m-1} a_i v_i(x)$$

while

$$C(x)=\operatorname{ch}(x+t_1+\cdots+t_{m-1})\prod_{i=1}^{m-1}\operatorname{sh}(x-t_i)$$
$$=\frac{\operatorname{ch}(mx)}{2^{m-1}}+\sum_{i=1}^{m-1} b_i v_i(x).$$

This implies

$$D(x)=2^{m-1}4^{r^2-r}\operatorname{ch}(x+t_1+\cdots+t_{m-1})$$
$$\times\prod_{1\leqslant i<j\leqslant m-1}\operatorname{sh}(t_j-t_i)\prod_{i=1}^{m-1}\operatorname{sh}(x-t_i),$$

and (3.5) follows upon setting $x=t_m$. The theorem is proved. ∎

The following lemma was used in the proof of Theorem 3.2:

LEMMA 3.3. *For any* $t_1<\cdots<t_{m-1}$,

$$\prod_{i=1}^{m-1}\operatorname{sh}(x-t_i)=\frac{\operatorname{sh}((m-1)x-t_1-\cdots-t_{m-1})}{2^{m-2}}+\sum_{i=1}^{m-2} a_i v_i(x),\quad m=2r,$$
$$=\frac{\operatorname{ch}((m-1)x-t_1-\cdots-t_{m-1})}{2^{m-2}}+\sum_{i=1}^{m-2} b_i u_i(x),\quad m=2r+1, \tag{3.6}$$

where $\{a_i\}_1^{m-2}$ *and* $\{b_i\}_1^{m-2}$ *are constants which depend on* m *and on the* $t_1,\dots, t_{m-1}$.

Proof. We repeatedly apply basic hyperbolic identities for products of sh and ch. ▮

We conclude this section with a result concerning a set of functions which we shall need later.

THEOREM 3.4. *Let* $m = 2r$. *Then the set of functions*

$$W_m = \{w_1,\dots, w_m\} = \{\operatorname{ch}(2x), \operatorname{sh}(2x),\dots, \operatorname{ch}(2rx), \operatorname{sh}(2rx)\} \tag{3.7}$$

is an Extended Complete Tschebyscheff system on $\mathbb{R}$.

Proof. As in the proof of Theorem 3.1, it suffices to check that each of the Wronskians $W(w_1,\dots, w_k)(x) > 0$ for $k = 1, 2,\dots, m$. Clearly $W(w_1)(x) = \operatorname{ch}(2x) > 0$ while $W(w_1, w_2)(x) = 2$. We now proceed by induction. If k is odd, say $k = 2n + 1$, then

$$W(w_1,\dots, w_k)(x) = \begin{vmatrix} \operatorname{ch}(2x) & \operatorname{sh}(2x) & \cdots & \operatorname{ch}((2n+2)\,x) \\ 2\ \operatorname{sh}(2x) & 2\operatorname{ch}(2x) & \cdots & (2n+2)\operatorname{sh}((2n+2)\,x) \\ \vdots & & & \\ 2^{2n}\operatorname{ch}(2x) & 2^{2n}\operatorname{sh}(2x) & \cdots & (2n+2)^{2n}\operatorname{ch}((2n+2)\,x) \end{vmatrix}$$

As in the proof of Theorem 3.1, we can now combine the odd rows to reduce W to a determinant with all zeros in the last row except for the element in the last column. Expanding by this row, we obtain

$$W(w_1,\dots, w_k)(x) = A\, W(w_1,\dots, w_{k-1})(x)\operatorname{ch}((2n+2)\,x),$$

where A is a positive constant.

The case where $k = 2n$ is even is similar. Here we must add combinations of the rows $2, 4,\dots, 2n$ together and combinations of the rows $1, 3,\dots, 2n-1$ together to reduce W to the form

$$W(w_1,\dots, w_k)(x) = B \left|\begin{array}{ccc|cc} & W(w_1,\dots, w_{k-2})(x) & & & \\ \hline 0 & \cdots & 0 & \operatorname{ch}(2nx) & \operatorname{sh}(2nx) \\ 0 & \cdots & 0 & \operatorname{sh}(2nx) & \operatorname{ch}(2nx) \end{array}\right|$$

with a positive constant B, (cf. the proof of Theorem 3.1). Then the result follows. ▮

4. Hyperbolic Divided Differences

In this section we define hyperbolic divided differences and establish several useful properties of them. Given any points $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_{m+1}$ and any sufficiently differentiable function f, we define the *mth order hyperbolic divided difference of f* by

$$\begin{aligned}[t_1,\dots,t_{m+1}]f &= 4^r D\begin{pmatrix} t_1,\dots,t_{m+1} \\ u_1,\dots,u_m,f \end{pmatrix} \Big/ D\begin{pmatrix} t_1,\dots,t_{m+1} \\ v_1,\dots,v_{m+1} \end{pmatrix}, \qquad m = 2^{r+1}, \\ &= 4^r D\begin{pmatrix} t_1,\dots,t_{m+1} \\ v_1,\dots,v_m,f \end{pmatrix} \Big/ D\begin{pmatrix} t_1,\dots,t_{m+1} \\ u_1,\dots,u_{m+1} \end{pmatrix}, \qquad m = 2r. \end{aligned} \tag{4.1}$$

This definition is well defined since by Theorem 3.1 the denominators in (4.1) can never be zero. The factor 4^r in the definition is a normalization factor.

The hyperbolic divided difference exhibits many of the same properties that the ordinary polynomial divided difference has (cf. [4, Sect. 2.7]). For example, if

$$t_1,\dots,t_{m+1} = \overbrace{\tau_1,\dots,\tau_1}^{l_1} < \cdots < \overbrace{\tau_d,\dots,\tau_d}^{l_d},$$

then

$$[t_1,\dots,t_{m+1}]f = \sum_{i=1}^{d} \sum_{j=1}^{l_i} \alpha_{ij} D^{j-1} f(\tau_i),$$

and thus the divided difference is a linear functional. It follows trivialy from the definition that it annihilates $\mathscr{H}_m$, i.e.,

$$[t_1,\dots,t_{m+1}]f = 0 \qquad \text{for all } \ f \in \mathscr{H}_m.$$

An important property of the ordinary divided differences is the fact that they are continuous functions of the points $t_1 \leqslant \cdots \leqslant t_{m+1}$, i.e.,

$$[t_{1,\varepsilon},\dots,t_{m+1,\varepsilon}]f \to [t_1,\dots,t_{m+1}]f \qquad \text{as} \quad \varepsilon \to 0$$

whenever $t_{i,\varepsilon} \to t_i$, $i = 1, 2,\dots, m+1$ (cf. [4, Theorem 2.53]). Since the hyperbolic divided differences are also defined by the ratio of two determinants, a similar proof serves to establish their continuity with respect to the location of the t's.

Our next theorem gives an explicit formula for the hyperbolic divided difference of a function in the case of distinct t's (cf. [4, Theorems 2.50 and 10.44] for the cases of ordinary and trigonometric divided differences).

THEOREM 4.1. *For any* $t_1 < t_2 < \cdots < t_{m+1}$,

$$[t_1, t_2, \ldots, t_{m+1}]f = \sum_{j=1}^{m+1} \left[f(t_j) \Big/ \prod_{\substack{i=1 \\ i \neq j}}^{m+1} \operatorname{sh}(t_j - t_i) \right]. \tag{4.2}$$

Proof. We simply expand the determinant in the numerator of the definition of the divided difference using Laplace's expansion on the last column, and then insert the exact values of the resulting determinants from Theorem 3.2. ∎

One of the important properties of the ordinary divided differences which we do not have in the case of hyperbolic divided differences is the Leibniz' rule (cf. [4, Theorem 2.52]). The following theorem is a useful substitute:

THEOREM 4.2. *Suppose* $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_{m+1}$ *with* $t_1 \neq t_{m+1}$. *Then for any function* f,

$$\begin{aligned}&[t_1, \ldots, t_{m+1}] \operatorname{sh}(x-y) f(y) \\ &\quad = \frac{-\operatorname{sh}(x-t_1)[t_1, \ldots, t_m]f - \operatorname{sh}(t_{m+1}-x)[t_2, \ldots, t_{m+1}]f}{\operatorname{sh}(t_{m+1}-t_1)}.\end{aligned} \tag{4.3}$$

Here the divided difference is taken with respect to the y *variable and* x *is any fixed real number. Similarly,*

$$\begin{aligned}&[t_1, \ldots, t_{m+1}] \operatorname{ch}(x-y) f(y) \\ &\quad = \frac{-\operatorname{ch}(x-t_1)[t_1, \ldots, t_m]f + \operatorname{ch}(t_{m+1}-x)[t_2, \ldots, t_{m+1}]f}{\operatorname{sh}(t_{m+1}-t_1)}.\end{aligned} \tag{4.4}$$

Proof. Using Theorem 4.1, we see that if $t_1 < t_2 < \cdots < t_{m+1}$, then

$$[t_1, \ldots, t_{m+1}] \operatorname{sh}(x-y) f(y) = \sum_{j=1}^{m+1} \left[f(t_j) \operatorname{sh}(x-t_j) \Big/ \sum_{\substack{i=1 \\ i \neq j}}^{m+1} \operatorname{sh}(t_j - t_i) \right],$$

$$-\operatorname{sh}(x-t_1)[t_1, \ldots, t_m]f = -\sum_{j=1}^{m} \left[f(t_j) \operatorname{sh}(x-t_1) \Big/ \prod_{\substack{i=1 \\ i \neq j}}^{m} \operatorname{sh}(t_j - t_i) \right],$$

and

$$-\operatorname{sh}(t_{m+1}-x)[t_2, \ldots, t_{m+1}]f = -\sum_{j=2}^{m+1} \left[f(t_j) \operatorname{sh}(t_{m+1}-x) \Big/ \prod_{\substack{i=2 \\ i \neq j}}^{m+1} \operatorname{sh}(t_j - t_i) \right].$$

To show the equality of the two sides of (4.3), it suffices to compare the coefficients of $f(t_1),\ldots,f(t_{m+1})$. It is easy to see that the coefficients of $f(t_1)$ and $f(t_{m+1})$ agree. But they also agree for all $1 < j < m+1$ since using the identity

$$\operatorname{sh}(a)\operatorname{sh}(b) = \tfrac{1}{2}(\operatorname{ch}(a+b) - \operatorname{ch}(a-b)),$$

we have

$$\operatorname{sh}(x-t_j)\operatorname{sh}(t_{m+1}-t_1) = -\operatorname{sh}(x-t_1)\operatorname{sh}(t_j-t_{m+1}) - \operatorname{sh}(t_{m+1}-x)\operatorname{sh}(t_j-t_1).$$

This shows that (4.3) holds for distinct t's. The fact that it is also valid for general $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_{m+1}$ now follows by the continuity of the divided differences.

The proof of (4.4) is similar. Suppose $t_1 < t_2 < \cdots < t_{m+1}$. Substituting in (4.4) from Theorem 4.1, we can again compare coefficients of $f(t_1),\ldots,f(t_{m+1})$. The fact that they agree for $1 < j < m+1$ follows in this case from the fact that

$$\operatorname{ch}(x-t_j)\operatorname{sh}(t_{m+1}-t_1) = -\operatorname{ch}(x-t_1)\operatorname{sh}(t_j-t_{m+1}) + \operatorname{ch}(t_{m+1}-x)\operatorname{sh}(t_j-t_1),$$

which in turn is easily checked using the simple identity

$$\operatorname{ch}(a)\operatorname{sh}(b) = \tfrac{1}{2}(\operatorname{sh}(a+b) - \operatorname{sh}(a-b)).$$

The result for arbitrary t's follows by the continuity of the divided differences. ∎

5. Hyperbolic B-Splines

Given a sequence of numbers

$$\cdots \leqslant y_{-1} \leqslant y_0 \leqslant y_1 \leqslant y_2 \leqslant \cdots$$

and integers i and $m > 0$, we define

$$\begin{aligned} Q_i^m(x) &= (-1)^m [y_i,\ldots,y_{i+m}]\operatorname{sh}(x-y)_+^{m+1}, &&\text{if } y_i < y_{i+m}, \\ &= 0, &&\text{otherwise.} \end{aligned} \tag{5.1}$$

We call Q_i^m the *mth order hyperbolic B-spline associated with the knots* $y_i,\ldots,y_{i+m}$.

For $m = 1$ and $y_i < y_{i+1}$ the hyperbolic B-spline is particularly simple; it is given by

$$\begin{aligned} Q_i^1(x) &= 1/\operatorname{sh}(y_{i+1}-y_i), &&y_i \leqslant x < y_{i+1}, \\ &= 0, &&\text{all other } x \end{aligned} \tag{5.2}$$

We can also give explicit expressions for Q_i^m in the cases when either y_i or y_{i+m} is an m-tuple knot.

THEOREM 5.1. *Suppose* $y_i < y_{i+1} = \cdots = y_{i+m}$. *Then*

$$\begin{aligned} Q_i^m(x) &= [\mathrm{sh}(x-y_i)]^{m-1}/[\mathrm{sh}(y_{i+m}-y_i)]^m, && y_i \leqslant x < y_{i+m}, \\ &= 0, && \textit{all other } x. \end{aligned} \tag{5.3}$$

Similarly, if $y_i = \cdots = y_{i+m-1} < y_{i+m}$, *then*

$$\begin{aligned} Q_i^m(x) &= [\mathrm{sh}(y_{i+m}-x)]^{m-1}/[\mathrm{sh}(y_{i+m}-y_i)]^m, && y_i \leqslant x < y_{i+m} \\ &= 0, && \textit{all other } x. \end{aligned} \tag{5.4}$$

Proof. These expressions follow by induction on m. The case $m=1$ is covered by (5.2). Now to get (5.3), for example, we apply (4.3) with $f(y) = [\mathrm{sh}(x-y)_+]^{m-2}$ which for $y_i \leqslant x < y_{i+m}$ yields

$$[y_i,\ldots,y_{i+m}](\mathrm{sh}(x-y))^{m-1} = \frac{-\mathrm{sh}(x-y_i)[y_i,\ldots,y_{i+m-1}](\mathrm{sh}(x-y))^{m-2}}{\mathrm{sh}(y_{i+m}-y_i)},$$

and the result for m follows from the result for $m-1$. The proof of (5.4) is similar. ∎

Theorem 5.2 describes the structure of Q_i^m for a general knot sequence, and identifies it as a hyperbolic spline.

THEOREM 5.2. *Let* $y_i < y_{i+m}$ *and suppose that*

$$y_i,\ldots,y_{i+m} = \overbrace{\tau_1,\ldots,\tau_1}^{l_1} < \cdots < \overbrace{\tau_d,\ldots,\tau_d}^{l_d}.$$

Then

$$Q_i^m(x) = \sum_{j=1}^{d} \sum_{k=1}^{l_j} \alpha_{jk} D_y^{k-1}[\mathrm{sh}(x-\tau_j)_+]^{m-1} \tag{5.5}$$

for some coefficients $\{\alpha_{jk}\}$. *Moreover,*

$$D_-^k Q_i^m(\tau_j) = D_+^k Q_i^m(\tau_j), \qquad k = 0, 1,\ldots, m-l_j-1, \quad j = 1, 2,\ldots, d. \tag{5.6}$$

Thus, Q_i^m *is a hyperbolic spline of order m with knots at the* $y_i,\ldots,y_{i+m}$.

Proof. Expansion (5.5) for Q_i^m follows directly from the expansion for the divided difference. The rest is elementary. ∎

The B-splines associated with knot sequences $y_i \leqslant y_{i+1} \leqslant \cdots \leqslant y_{i+m}$ where some of the y's are equal to others can be regarded as arising as limits of B-splines associated with distinct knots. In particular, it is easy to show from the continuity of divided differences that:

if $Q^m_{i,\nu}(x)$ is the B-spline associated with $y^\nu_i \leqslant y^\nu_{i+1} \leqslant \cdots \leqslant y^\nu_{i+m}$

and

$Q^m_i(x)$ is the B-spline associated with $y_i \leqslant y_{i+1} \leqslant \cdots \leqslant y_{i+m}$,

then

$$y^\nu_j \to y_j, \qquad j = i, i+1, \ldots, i+m \quad \text{as} \quad \nu \to \infty$$

implies

$$D^k_+ Q^m_{i,\nu}(x) \to D^k_+ Q^m_i(x) \qquad \text{for all } x \in \mathbb{R} \backslash J^k_i,$$

where

$$J^k_i = \{y_j : y_j \text{ is a knot of } Q^m_i \text{ of multiplicity } m-k \text{ or more}\}. \tag{5.7}$$

(For the details of the proof in the polynomial spline case, see [4, Theorem 4.26]).

One of the most important properties of polynomial B-splines is the fact that they can be computed by a convenient recurrence relation. The following theorem gives the analog for hyperbolic splines:

Theorem 5.3. *Let $m \geqslant 2$ and suppose that $y_i < y_{i+m}$. Then for all $x \in \mathbb{R}$,*

$$Q^m_i(x) = \frac{\operatorname{sh}(x - y_i)\, Q^{m-1}_i(x) + \operatorname{sh}(y_{i+m} - x)\, Q^{m-1}_{i+1}(x)}{\operatorname{sh}(y_{i+m} - y_i)}. \tag{5.8}$$

Proof. For $y_i < y_{i+1} = \cdots = y_{i+m}$ or $y_i = \cdots = y_{i+m-1} < y_{i+m}$, the result follows directly from Theorem 5.1. Thus we may assume that $y_{i+1} < y_{i+m}$ and $y_i < y_{i+m-1}$. Now since

$$[\operatorname{sh}(x-y)_+]^{m-1} = \operatorname{sh}(x-y)[\operatorname{sh}(x-y)_+]^{m-2},$$

if we apply $(-1)^m [y_i, \ldots, y_{i+m}]$ to both sides and use identity (4.3), we obtain (5.8). ∎

We give a number of applications of this result in the following sections. The next theorem gives a similar recursion for the derivative of a hyperbolic B-spline.

THEOREM 5.4. *Let $m \geqslant 2$ and suppose that $y_i < y_{i+m}$. Then for all $x \in \mathbb{R}$,*

$$DQ_i^m(x) = (m-1)\frac{\operatorname{ch}(x-y_i)\, Q_i^{m-1}(x) - \operatorname{ch}(y_{i+m}-x)\, Q_{i+1}^{m-1}(x)}{\operatorname{sh}(y_{i+m}-y_i)}. \quad (5.9)$$

Proof. The assertion follows immediately upon application of $(-1)^m[y_i,\ldots,y_{i+m}]$ to both sides of the identity

$$D_x[\operatorname{sh}(x-y)_+]^{m-1} = (m-1)\operatorname{ch}(x-y)[\operatorname{sh}(x-y)_+]^{m-2}$$

along with the use of identity (4.4). ∎

6. More on Hyperbolic B-Splines

With recurrence relation (5.8) at our disposal, we can now give a very precise result about the shape of Q_i^m.

THEOREM 6.1. *Let $m > 1$ and suppose $y_i < y_{i+m}$. Then*

$$Q_i^m(x) > 0 \qquad \text{for} \quad y_i < x < y_{i+m}, \quad (6.1)$$

$$Q_i^m(x) = 0 \qquad \text{for} \quad x < y_i \quad \text{and} \quad y_{i+m} < x. \quad (6.2)$$

At the endpoints of the interval (y_i, y_{i+m}) we have

$$\begin{aligned}(-1)^{k+m-\alpha_i} D_+^k Q_i^m(y_i) &= 0, \qquad k = 0, 1,\ldots, m-1-\alpha_i,\\ &> 0, \qquad k = m-\alpha_i,\ldots, m-1;\end{aligned} \quad (6.3)$$

and

$$\begin{aligned}(-1)^{m-\beta_{i+m}} D_-^k Q_i^m(y_{i+m}) &= 0, \qquad k = 0, 1,\ldots, m-1-\beta_{i+m},\\ &> 0, \qquad k = m-\beta_{i+m},\ldots, m-1;\end{aligned} \quad (6.4)$$

where

$$\alpha_i = \max\{j : y_i = \cdots = y_{i+j-1}\},$$

$$\beta_{i+m} = \max\{j : y_{i+m} = \cdots = y_{i+m-j+1}\}.$$

The quantity α_i tells how many of the points $y_i \leqslant \cdots \leqslant y_{i+m}$ are equal to y_i, while β_{i+m} tells how many of them are equal to y_{i+m}.

Proof. Property (6.1) follows by induction, while (6.2) comes directly from the definition. The sign properties of the derivatives (6.3)–(6.4) are also established by induction. ∎

Our next theorem shows that the hyperbolic B-spline is the kernel in a Peano representation of hyperbolic divided differences.

THEOREM 6.2. *For any* $f \in L_1^m[y_i, y_{i+m}]$,

$$[y_i,\dots,y_{i+m}]f = \int_{y_i}^{y_{i+m}} \frac{Q_i^m(y)\, L_m f(y)\, dy}{(m-1)!}. \tag{6.5}$$

Proof. If we apply the divided difference to the Taylor expansion given in (2.6), we obtain

$$[y_i,\dots,y_{i+m}]f = \int_{y_i}^{y_{i+m}} \frac{\tilde{Q}_i^m(y)\, L_m f(y)\, dy}{(m-1)!},$$

where

$$\tilde{Q}_i^m(y) = [y_i,\dots,y_{i+m}]_x \operatorname{sh}(x-y)_+^{m-1}.$$

Now since for all x and y

$$\operatorname{sh}(x-y)_+^{m-1} - (-1)^m \operatorname{sh}(y-x)_+^{m-1} = \operatorname{sh}(x-y)^{m-1} \in \mathscr{H}_m,$$

applying $[y_i,\dots,y_{i+m}]_x$, we conclude that

$$\tilde{Q}_i^m(y) - Q_i^m(y) = 0, \qquad \text{all} \quad y \in \mathbb{R}\backslash J_i^m$$

(cf. (5.7)), and (6.5) follows. ∎

The only difference between $Q_i^m(y)$ and $\tilde{Q}_i^m(y)$ is that the first is left continuous while the second is right continuous. This makes no difference except at an m-tuple knot.

Theorem 6.2 can be used to compute the integrals of the B-splines. We have

THEOREM 6.3. *Let* $m \geqslant 1$ *and* $y_i \leqslant y_{i+1} \leqslant \cdots \leqslant y_{i+m}$ *be given. Then*

$$\begin{aligned}\int_{y_i}^{y_{i+m}} Q_i^m(x)\, dx &= (m-1)!\,(-1)^r \left[\frac{2r!}{(2r)!}\right]^2 [y_i,\dots,y_{i+m}]\, 1, \qquad m = 2r,\\ &= (m-1)!\,(-1)^r \frac{1}{[2^r r!]^2} [y_i,\dots,y_{i+m}]\, x, \qquad m = 2r+1.\end{aligned} \tag{6.6}$$

Proof. It is easily checked that

$$L_m 1 = (-1)^r \left[\frac{(2r)!}{2r!}\right]^2 \qquad \text{if} \quad m = 2r$$

while

$$L_m x = (-1)^r\,[2^r r!]^2 \qquad \text{if} \quad m = 2r+1.$$

Substituting the function $f = 1$ (if m is even) or $f = x$ (if m is odd) in (6.5) leads immediately to (6.6). ∎

We now turn to some properties of the normalized hyperbolic B-splines defined by

$$N_i^m(x) = \operatorname{sh}(y_{i+m} - y_i)\, Q_i^m(x). \tag{6.7}$$

For $m = 1$, the normalized B-spline associated with the knots $y_i < y_{i+1}$ is given by

$$\begin{aligned} N_i^1(x) &= 1, \qquad y_i \leqslant x < y_{i+1} \\ &= 0, \qquad \text{all other } x. \end{aligned} \tag{6.8}$$

We can give an expansion of the kernel $[\operatorname{sh}(y-x)]^{m-1}$ in terms of the normalized B-splines.

THEOREM 6.4. *Let $l \leqslant r$ and $y_l < y_{r+1}$. Then for any $y \in \mathbb{R}$,*

$$[\operatorname{sh}(y-x)]^{m-1} = \sum_{i=l+1-m}^{r} \phi_{i,m}(y)\, N_i^m(x), \qquad \text{all} \quad y_l \leqslant x < y_{r+1}, \tag{6.9}$$

where

$$\phi_{i,m}(y) = \prod_{v=1}^{m-1} \operatorname{sh}(y - y_{i+v}).$$

Proof. We proceed by induction. For $m = 1$ the result follows from (6.8) and asserts that

$$1 \equiv \sum_{i=l+1-m}^{r} N_i^1(x).$$

Now assuming the identity has been established for $m-1$, we have

$$\sum_{i=l+1-m}^{r} \phi_{i,m}(y)\, N_i^m(x)$$
$$= \sum_{i=l+1-m}^{r} \phi_{i,m}(y)[\operatorname{sh}(x - y_i)\, Q_i^{m-1}(x) + \operatorname{sh}(y_{i+m} - x)\, Q_{i+1}^{m-1}(x)]$$

$$= \sum_{i=l+1-m}^{r} Q_i^{m-1}(x)[\mathrm{sh}(x-y_i)\,\phi_{i,m}(y) + \mathrm{sh}(y_{i+m-1}-x)\,\phi_{i-1,m}(y)]$$

$$= \sum_{i=l+1-m}^{r} Q_i^{m-1}(x)\,\phi_{i,m-1}(y)[\mathrm{sh}(x-y_i)\,\mathrm{sh}(y-y_{i+m-1})$$

$$+ \mathrm{sh}(y_{i+m-1}-x)\,\mathrm{sh}(y-y_i)].$$

Using elementary identities on the hyperbolic functions, we see that the quantity in the square brackets is $[\ \ ] = \mathrm{sh}(y-x)\,\mathrm{sh}(y_{i+m-1}-y_i)$, and we get

$$\sum_{i=l+1-m}^{r} \phi_{i,m}(y)\,N_i^m(x) = \mathrm{sh}(y-x) \sum_{i=l+1-m}^{r} \phi_{i,m-1}(y)\,N_i^{m-1}(x)$$

$$= \mathrm{sh}(y-x)[\mathrm{sh}(y-x)]^{m-2} = [\mathrm{sh}(y-x)]^{m-1}. \quad \blacksquare$$

7. A Partition of Unity

One of the most interesting facts about polynomial B-splines is that they can be used to give a partition of unity. The following theorem is the hyperbolic analog of this result. Note, however, that we only assert the existence of such a partition for the case of m odd. Indeed, when m is even, the space of hyperbolic splines does not even contain the function 1.

THEOREM 7.1. *Let $m = 2r+1$. Then for all $y_l \leqslant x < y_r$,*

$$1 = \sum_{l+1-m}^{r} \alpha_i^m N_i^m(x), \tag{7.1}$$

where

$$\alpha_i^m = (-1)^r\, 2^{2r-1} a_1^i \bigg/ \binom{2r-1}{r} > 0 \tag{7.2}$$

and a_1^i is the constant term in the expansion (cf. Lemma 3.3)

$$\phi_{i,m}(y) = \prod_{\nu=1}^{m-1} \mathrm{sh}(y-y_{i+\nu}) = \sum_{j=1}^{m} a_j^i u_j(y).$$

Proof. We apply the operator

$$M = (D_y^2 - (2r)^2) \cdots (D_y^2 - 2^2)$$

to both sides of the Marsden identity (6.9). Since M annihilates the functions $\mathrm{ch}(2y), \mathrm{sh}(2y),\dots, \mathrm{ch}(2ry), \mathrm{sh}(2ry)$, it follows that

$$M[\mathrm{sh}(y-x)]^{m-1} = M(-1)^r \binom{2r-1}{r} \Big/ 2^{2r-1}$$

while

$$M\phi_{i,m}(y) = Ma_1^i.$$

Identity (7.1) follows.

It remains to show that $\alpha_i^m > 0$, or equivalently, that $(-1)^r a_1^i > 0$. The coefficients of $\phi_{i,m}(y)$ must satisfy the system of equations

$$\begin{bmatrix} \phi_{i,m}(y_{i+1}) \\ \vdots \\ \phi_{i,m}(y_{i+m-1}) \\ \phi_{i,m}(y_{i+m}) \end{bmatrix} = \begin{bmatrix} u_1(y_{i+1}) & \cdots & u_m(y_{i+1}) \\ \vdots & & \\ u_1(y_{i+m-1}) & \cdots & u_m(y_{i+m-1}) \\ u_1(y_{i+m}) & \cdots & u_m(y_{i+m}) \end{bmatrix} \begin{bmatrix} a_1^i \\ \vdots \\ a_{m-1}^i \\ a_m^i \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ A \end{bmatrix},$$

where $A = \prod_{\nu=1}^{m-1} \mathrm{sh}(y_{i+m} - y_{i+\nu}) > 0$. But then Cramer's rule shows that

$$a_1^i = AD \begin{pmatrix} y_{i+1},\dots, y_{i+m-1} \\ u_2,\dots, u_m \end{pmatrix} = A(-1)^r D \begin{pmatrix} y_{i+1},\dots, y_{i+m-1} \\ w_1,\dots, w_{m-1} \end{pmatrix} > 0,$$

where the last inequality follows from Theorem 3.4. (Note that $\{u_2, u_3,\dots, u_m\} = \{w_2, w_1,\dots, w_{m-1}, w_{m-2}\}$, which accounts for the $(-1)^r$ in the above string of equalities.) ∎

Theorem 7.1 is not quite of the same form as its analog for polynomial splines where the B-splines themselves form a partition of unity without the factors α_i^m. It is possible to derive explicit formulae for the α_i^m, at least for small m.

EXAMPLE 7.2. The hyperbolic B-splines of order 1 and 3 satisfy the relations

$$1 = \sum_{i=l+1-m}^{r} N_i^1(x), \tag{7.3}$$

$$1 = \sum_{i=l+1-m}^{r} \mathrm{ch}(y_{i+2} - y_{i+1})\, N_i^3(x). \tag{7.4}$$

Proof. Partition (7.3) is trivial in veiw of the definition of the N_i^1's. Partition (7.4) follows from Theorem 7.1 and the fact that

$$\phi_{i,3}(y) = \operatorname{sh}(y-y_{i+1})\operatorname{sh}(y-y_{i+2}) = \frac{\operatorname{ch}(2y-y_{i+1}-y_{i+2})-\operatorname{ch}(y_{i+2}-y_{i+1})}{2}$$

$$= \frac{\operatorname{ch}(2y)\operatorname{ch}(y_{i+1}+y_{i+2})-\operatorname{sh}(2y)\operatorname{sh}(y_{i+1}+y_{i+2})-\operatorname{ch}(y_{i+2}-y_{i+1})}{2}.$$

∎

Identity (6.9) can be used to give a variety of identities involving the normalized B-splines. For example, we have the following somewhat curious result:

THEOREM 7.3. *For any m and all $y_{l+m} \leqslant x \leqslant y_r$,*

$$1 = \sum_{i=l}^{r} \operatorname{ch}((m-1)x - y_{i+1} - \cdots - y_{i+m-1})\, N_i^m(x).$$

Proof. This result can be established by applying an appropriate operator to both sides of (6.9). Or, it can be established by induction. It is trivially true for $m = 1$. Now using recurrence relation (5.8), we note that (with $m_1 = m-1$ and $\beta_i^m = y_{i+1} + \cdots + y_{i+m-1}$),

$$\begin{aligned}
&\sum \operatorname{ch}(m_1 x - \beta_i^m)\, N_i^m(x)\\
&\quad = \sum \operatorname{ch}(m_1 x - \beta_i^m)[Q_i^{m-1}(x)\operatorname{sh}(x-y_i) + \operatorname{sh}(y_{i+m}-x)\, Q_{i+1}^{m-1}(x)]\\
&\quad = \sum Q_i^{m-1}(x)[\operatorname{ch}(m_1 x - \beta_i^m)\operatorname{sh}(x-y_i)\\
&\qquad + \operatorname{ch}(m_1 x - \beta_{i-1}^m)\operatorname{sh}(y_{i+m-1}-x)].
\end{aligned}$$

But

$$\begin{aligned}
&[\operatorname{ch}(m_1 x - \beta_i^m)\operatorname{sh}(x-y_i) + \operatorname{ch}(m_1 x - \beta_{i-1}^m)\operatorname{sh}(y_{i+m-1}-x)]\\
&\quad = \operatorname{ch}(m_2 x - \beta_i^{m-1})\operatorname{sh}(y_{i+m-1}-y_i),
\end{aligned}$$

where $m_2 = m-2$ and $\beta_i^{m-1} = y_{i+1} + \cdots + y_{i+m-2}$. It follows that

$$\sum \operatorname{ch}(m_1 x - \beta_i^m)\, N_i^m(x) = \sum \operatorname{ch}(m_2 x - \beta_i^{m-1})\, N_i^{m-1}(x) = 1$$

by the inductive hypothesis. ∎

8. A Basis and a Dual Basis

We are now in a position to describe a basis of B-splines for the space of hyperbolic splines $\mathscr{S}(\mathscr{H}_m; \mathscr{M}; \Delta)$. Following the construction for polynomial splines (cf. [4]), suppose $y_1 \leqslant y_2 \leqslant \cdots \leqslant y_{2m+K}$ is a set of points with

$$y_1 \leqslant y_2 \leqslant \cdots \leqslant y_m \leqslant a \qquad \text{and} \qquad b \leqslant y_{m+K+1} \leqslant \cdots \leqslant y_{2m+K}$$

and

$$y_{m+1} \leqslant \cdots \leqslant y_{m+K} = \overbrace{x_1, \ldots, x_1}^{m_1}, \ldots, \overbrace{x_k, \ldots, x_k}^{m_k}.$$

Let

$$B_i(x) = N_i^m(x), \qquad i = 1, 2, \ldots, m+K, \tag{8.1}$$

where the N's are the normalized B-splines defined in (6.7). (In the case where $b = y_{m+K+1} = \cdots = y_{2m+K}$, we modify B_{m+K} slightly by taking $B_{m+K}(b) = B_{m+K}(b-)$).

THEOREM 8.1. *The functions* $\{B_i(x)\}_1^{m+K}$ *form a basis for* $\mathscr{S}(\mathscr{H}_m; \mathscr{M}; \Delta)$.

Proof. It is clear from our earlier results on B-splines that each B_i is an element of $\mathscr{S}$. (The special definition of B_{m+K} at b was necessary to insure this—cf. the discussion in [4, p. 117].) The fact that these functions are linearly independent follows immediately from Theorem 8.2 below, and since $\mathscr{S}$ has dimension $m+K$, the assertion is proved. ∎

In order to establish the linear independence of the hyperbolic B-splines, we now construct a dual basis of linear functionals $\{\lambda_i\}_1^{m+K}$. Again, we follow the construction in the polynomial spline case (cf. [4, pp. 145 and following]). For each $j = 1, 2, \ldots, m+K$, let $G_j(x)$ be the transition function defined in the proof of [4, Theorem 4.41], and let $\phi_{jm}(y)$ be the function defined in Theorem 8.1 above. Then for any sufficiently smooth function f, we define

$$\lambda_j f = \int_{y_j}^{y_{j+m}} \frac{f(x)\, L_m \Psi_j(x)\, dx}{(m-1)!\,(y_{j+m} - y_j)}, \qquad j = 1, 2, \ldots, m+K, \tag{8.2}$$

where L_m is the differential operator defined in (1.1) and where

$$\Psi_j(x) = G_j(x)\, \phi_{j,m}(x).$$

THEOREM 8.2. *The linear functionals* $\{\lambda_i\}_1^{m+K}$ *form a dual basis for* $\{B_i\}_1^{m+K}$, *i.e.*,

$$\begin{aligned} \lambda_j B_i = \delta_{ij} &= 1, \qquad \text{if} \quad i=j, \\ &= 0, \qquad \text{otherwise}, \end{aligned} \tag{8.3}$$

for all $1 \leqslant i,j \leqslant m+K$.

Proof. By the Peano representation for divided differences given in (6.5), we have

$$\lambda_j B_i = \int_{y_j}^{y_{j+m}} \frac{B_i(x)\, L_m \Psi_j(x)\, dx}{(m-1)!\,(y_{m+j}-y_j)} = \left(\frac{y_{i+m}-y_i}{y_{j+m}-y_j}\right) [y_i,\ldots,y_{i+m}]\, \Psi_j,$$

all $1 \leqslant i,\ j \leqslant m+K$. Now if $i>j$, this is zero since $\Psi_j \in \mathscr{H}_m$ and hence is annihilated by the divided difference. If $i<j$, then we again get zero since Ψ_j agrees with the function 0 on the points $y_i,\ldots,y_{i+m}$. Finally, if $i=j$, we note that Ψ_j agrees with $\operatorname{sh}(y-y_i)\,\phi_{i,m}(y)$ on $y_i,\ldots,y_{i+m}$, and so

$$\lambda_i B_i = [y_i,\ldots,y_{i+m}] \prod_{\nu=i}^{i+m+1} \operatorname{sh}(y-y_\nu),$$

which is easily shown by induction to have the value 1. Indeed, for $m=1$ we have $[y_i,y_{i+1}]\operatorname{sh}(y-y_i)=1$. Now assuming the result for $m-1$, using recursion formula (4.3), we have

$$\begin{aligned} &[y_i,\ldots,y_{i+m}]\operatorname{sh}(y-y_i) \prod_{\nu=i+1}^{i+m-1} \operatorname{sh}(y-y_\nu) \\ &\quad = [y_{i+1},\ldots,y_{i+m}] \prod_{\nu=i+1}^{i+m-1} \operatorname{sh}(y-y_\nu) = 1. \qquad \blacksquare \end{aligned}$$

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